

I A-automorphisms of various classes of groups

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- **Definition 1.** The *lower central series* of a group G is

$$\gamma_1(G) = G \geq \gamma_2(G) \geq \cdots \gamma_i(G) \geq \cdots$$

where $\gamma_2(G) = G' = [G, G]$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$.

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where $\gamma_2(G) = G' = [G, G]$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$.

- **Definition 2.** A group G is termed *nilpotent* of class c if $\gamma_{c+1}(G) = 1$ for some c and $\gamma_c(G) \neq 1$.

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- The group of *IA-automorphisms* of G is $\ker(\text{Aut}(G) \mapsto \text{Aut}(G/G'))$. Thus, $\text{IA}(G) = \{\alpha \in \text{Aut}(G) : g^{-1}(g\alpha) \in G' (g \in G)\}$.

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- $\text{Inn}(G) \leq \text{IA}(G)$ for any group G .
- $G/\zeta(G) \cong \text{Inn}(G)$.

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 - If G is finitely generated, $IA(G)$ is finitely generated.
 - If G is torsion-free, $IA(G)$ is torsion-free.

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- **Fundamental Theorem of Localization.** Given a nilpotent group G there exists a localization map $G \rightarrow G_{(p)}$ where $G_{(p)}$ is p -local and nilpotent.
- A group G is *metabelian* if $G' = [G, G]$ is abelian.
- **Theorem [MZ]** If the p -localizations of two finitely generated nilpotent, torsion-free and metabelian groups are isomorphic, then the p -localizations of their IA -groups are also isomorphic.

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 - ③ If n is not square-free then $IA(G)$ is not finitely generated.

I A-automorphisms of groups with constant upper central series (joint work with M.H. Dean and M. Bonanome)

- *Central automorphisms*

$$\text{Aut}_c(G) = \{ \alpha \in \text{Aut}(G) : g^{-1}(g\alpha) \in \zeta(G) \ (g \in G) \}.$$

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- The upper central series of G

$$1 = Z_0 \trianglelefteq Z_1 \trianglelefteq \dots \trianglelefteq Z_i \trianglelefteq \dots$$

is defined recursively for $i > 0$ by the condition

$$Z_{i+1}/Z_i = \zeta(G/Z_i).$$

Lemma

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- (ii) $\zeta(G/\zeta(G)) = 1$.
- (iii) $\zeta(G) = Z_i$ for all $i = 1, 2, \dots$
- (iv) $\text{Inn}(G)$ has trivial center.

- G will be termed an \mathcal{H} -group if $Z_1 = Z_2$ in the upper central series.

Lemma

Let G be an \mathcal{H} -group. Then

$$\text{gp}(\text{Inn}(G), \text{Aut}_c(G)) = \text{Inn}(G) \times \text{Aut}_c(G).$$

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- $\text{Inn}(G) \cap \text{Aut}_c(G) \leq \zeta(\text{Inn}(G)) = 1$.



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Lemma

Let G be an \mathcal{H} -group such that $G' \geq \zeta(G)$ and $IA(G/\zeta(G)) = Inn(G/\zeta(G))$. Then $Inn(G)$ is a complete set of representatives of $Aut_c(G)$ in $IA(G)$.

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There exists $x \in G$ such that

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\bar{x} induces the desired inner automorphism. □

Theorem

Let G be a group such that $Z_1 = Z_2$ in the upper central series and $\zeta(G) \leq G'$. If $IA(G/\zeta(G)) = Inn(G/\zeta(G))$, then

$$IA(G) = Inn(G) \times Aut_c(G).$$

A useful lemma

Lemma

Let $G = \text{gp}(x_1, \dots, x_n)$ be a finitely generated \mathcal{H} -group such that $\zeta(G) \leq G'$. Suppose that there exists a presentation for G in which every relator is a product of commutators. If $\text{Aut}_c(G)$ is finitely generated, then $\zeta(G)$ is finitely generated.

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Proof.

Suppose $\{\alpha_1, \dots, \alpha_m\}$ generates $Aut_c(G)$. Then

$$x_i \alpha_j = x_i z_{ij}$$

with $i = 1, \dots, n$; $j = 1, \dots, m$; and $z_{ij} \in \zeta(G)$. Then the z_{ij} generate the $\zeta(G)$. □

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- $IA(G)$ is not finitely generated.

Thank you!